

L02 Games: Definitions and Existence of Nash Equilibrium

CS 280 Introduction to Algorithmic Game Theory

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Definitions

Definition (**Normal Form Games**). *A normal form game is specified by*

- *set of n players $[n] = \{1, \dots, n\}$*
- *For each player i a set of strategies/actions S_i and a utility $u_i : \times_{j=1}^n S_j \rightarrow \mathbb{R}$ denoting the payoff of i .*
- *set of strategy profiles $S = S_1 \times \dots \times S_n$.*

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- *set of strategy profiles $S = S_1 \times \dots \times S_n$.*

Example (Rock-Paper-Scissors). *We have that:*

- $n = 2$
- $S_1, S_2 = \{R, P, S\}$.
- $u_1(R, R) = 0, u_1(R, P) = -1, u_1(R, S) = 1, u_1(P, R) = 1, u_1(P, P) = 0, u_1(P, S) = -1, u_1(S, R) = -1, u_1(S, P) = 1, u_1(S, S) = 0.$
- $u_2 = -u_1$

Definitions

Definition (Mixed strategies). *The set of mixed strategies available to player i are all distributions over S_i*

$$\Delta_i = \{x_i : \sum_{s_i \in S_i} x_i(s_i) = 1 \text{ and } x_i \geq \mathbf{0}\}$$

We also denote $\Delta = \Delta_1 \times \dots \times \Delta_n$ the set of mixed strategies of all players and Δ_{-i} the set of mixed strategies of all players excluding i .

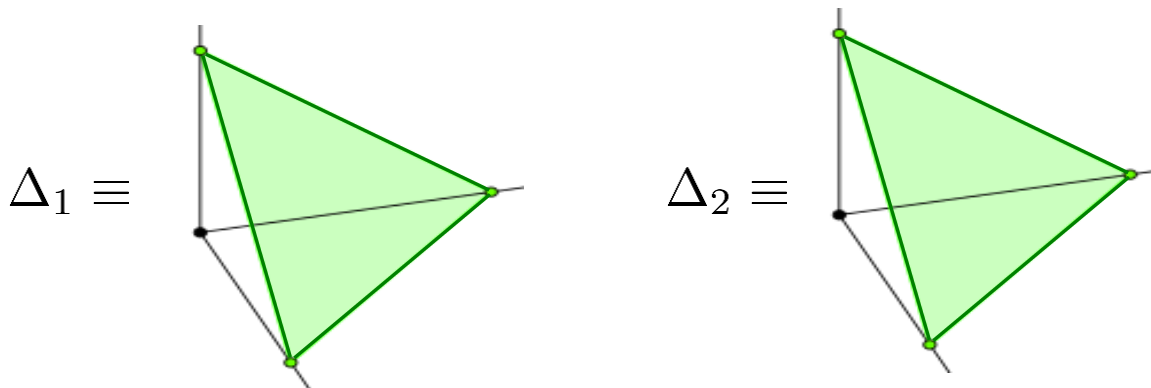
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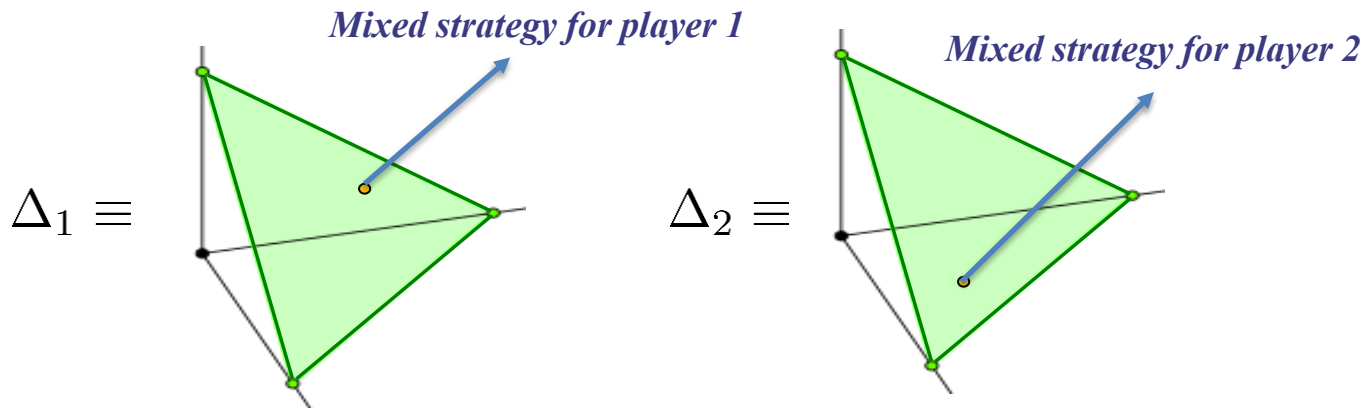
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Definitions

Definition (Expected utility). *Given a mixed strategy $x \in \Delta$, the expected utility of player i is*

$$u_i(x) = \sum_{(s_1, \dots, s_n) \in S} u_i(s_1, \dots, s_n) \prod_{j=1}^n x_j(s_j)$$

or (in a more compact way)

$$u_i(x) = \mathbb{E}_{s \sim x} u_i(s).$$

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Example (Rock-Paper-Scissors). We have that:

- Say $x_1 = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$ and $x_2 = (\frac{1}{2}, \frac{1}{2}, 0)$.
- Then $u_1(x_1, x_2) = \frac{3}{12}u_1(R, R) + \frac{3}{12}u_1(R, P) + 0u_1(R, S) + \frac{2}{12}u_1(P, R) + \frac{2}{12}u_1(P, P) + 0u_1(P, S) + \frac{1}{12}u_1(S, R) + \frac{1}{12}u_1(S, P) + 0u_1(S, S) = -\frac{3}{12} + \frac{2}{12} - \frac{1}{12} + \frac{1}{12} = -\frac{1}{12}$
- $u_2(x_1, x_2) = \frac{1}{12}$

Existence of Nash Equilibrium

Definition (Nash equilibrium). *A mixed strategy $x \equiv (x_1; \dots; x_n) \in \Delta$ is a Nash equilibrium if and only if for all agents i and $x'_i \in \Delta_i$ we have*

$$u_i(x_i; x_{-i}) \geq u_i(x'_i; x_{-i}).$$

Definition (ϵ -approximate Nash equilibrium). *A mixed strategy $x \equiv (x_1; \dots; x_n) \in \Delta$ is a ϵ -approximate Nash equilibrium if and only if for all agents i and $x'_i \in \Delta_i$ we have*

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Theorem (Nash 51'). Every game with a finite number of players and actions *has a Nash equilibrium*.

Proof of existence of NE

Before we proceed with the proof, an important theorem from topology...

Theorem (Brouwer). *Let D be a nonempty, convex, compact subset of $\mathbb{R}^d$ and $f : D \rightarrow D$ a *continuous function*. There always exists $x \in D$ such that*

$$f(x) = x.$$

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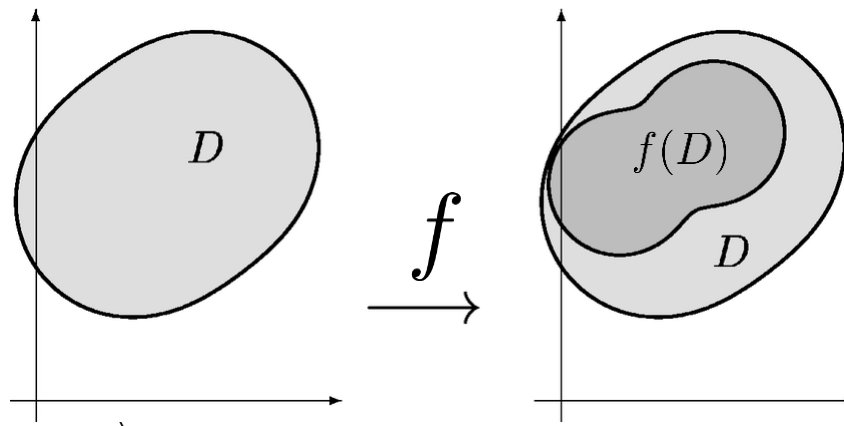
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Remark:

x is called a **fixed point** of f . Uniqueness is not true!



Proof of existence of NE

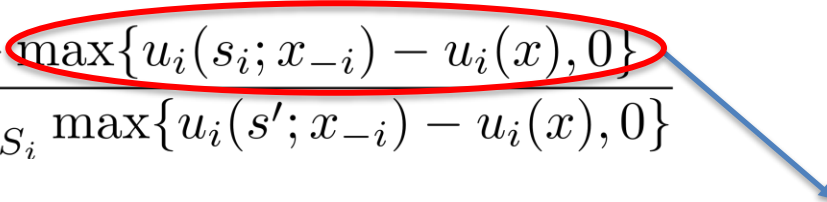
Proof. Consider any finite game. Define function $f : \Delta \rightarrow \Delta$ as follows

$$f_{is_i}(x) = \frac{x_i(s_i) + \max\{u_i(s_i; x_{-i}) - u_i(x), 0\}}{1 + \sum_{s' \in S_i} \max\{u_i(s'; x_{-i}) - u_i(x), 0\}}$$

Gain if i were to switch to pure strategy s_i instead of mixed x_i

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For RPS

$$f(x) = (f_{1R}(x), f_{1P}(x), f_{1S}(x), f_{2R}(x), f_{2P}(x), f_{2S}(x))$$

with $f_{1R}(x) = \left(\frac{x_{1R} + \max\{-x_{2P} + x_{2S} - u_1(x), 0\}}{1 + \max\{-x_{2P} + x_{2S} - u_1(x), 0\} + \max\{x_{2R} - x_{2S} - u_1(x), 0\} + \max\{-x_{2R} + x_{2P} - u_1(x), 0\}} \right)$
etc...

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Observations:

- For each player $i \Rightarrow \sum_{s' \in S_i} f_{is'}(x) = 1$. why?

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Fixed point always exists!!

Proof of existence of NE

Proof cont.

$$f_{is_i}(x) = \frac{x_i(s_i) + \max\{u_i(s_i; x_{-i}) - u_i(x), 0\}}{1 + \sum_{s' \in S_i} \max\{u_i(s'; x_{-i}) - u_i(x), 0\}}$$

Let x^* be a fixed point of f . We will show that x^* is a Nash Equilibrium!

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Since $f(x^*) = x^*$ it holds that for all agents i and $s \in S_i$ that $f_{is}(x^*) = x_i^*(s)$

$$\Rightarrow x_i^*(s) \sum_{s' \in S_i} \max\{u_i(s'; x_{-i}^*) - u_i(x^*), 0\} = \max\{u_i(s; x_{-i}^*) - u_i(x^*), 0\}$$

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Cases:

- $x_i^*(s) = 0 \Rightarrow u_i(s; x_{-i}^*) \leq u_i(x^*)$.

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Contradiction!

Proof of existence of NE

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- $x_i^*(s) > 0$ then if $u_i(s; x_{-i}^*) \geq u_i(x^*)$.

In particular since $u_i(x^*) = \sum_{s'} u_i(s'; x_{-i}^*) x_i^*(s')$ we have $u_i(s; x_{-i}^*) = u_i(x^*)$ whenever $x_i^*(s) > 0$.

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It holds for all agents i :

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Consider any $\tilde{x}_i$, we will show that

$$u_i(x_i^*; x_{-i}^*) \geq u_i(\tilde{x}_i; x_{-i}^*).$$

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From two bullets we get $\tilde{x}_i(s)u_i(s, x_{-i}^*) \leq \tilde{x}_i(s)u_i(x^*)$ so taking the summation

$$u_i(\tilde{x}_i, x_{-i}^*) = \sum_{s'} \tilde{x}_i(s')u_i(s'; x_{-i}^*) \leq \sum_{s'} \tilde{x}_i(s')u_i(x^*) = u_i(x^*)$$

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**So finding Nash Equilibria is like computing fixed points!
Can it be computationally hard? YES...**

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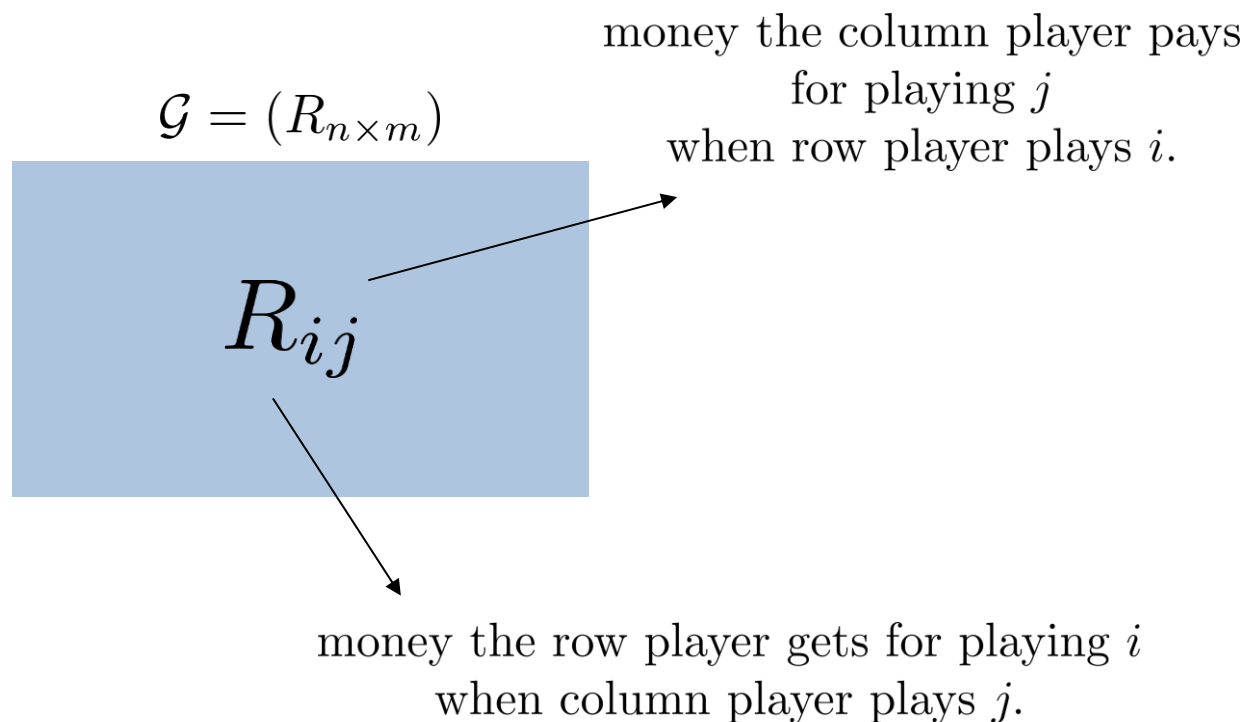
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Zero-sum Games

- 2 players: **Row** and **Column**
- n, m **strategies** available
- **Payoff** matrix R of size $n \times m$.



Zero-sum Games

Column player
chooses $y \in \Delta_m$

Row player
chooses $x \in \Delta_n$

R_{ij}



Row gets $x^\top Ry$.
Column pays $x^\top Ry$.

Example: Two candidates are aiming for presidency.

	Tax-cuts	Society
Economy	3, -3	-1, 1
Education	-2, 2	1, -1

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**How should
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Suppose row player plays (x_{11}, x_{12}) . How should column player respond?

Zero-sum Games

Example: Two candidates are aiming for presidency.

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Suppose row player plays (x_{11}, x_{12}) . How should column player respond?

Answer: If she chooses Tax-cuts she gets in expectation $u_2(x_1, 'Tax - cuts') = -3x_{11} + 2x_{12}$ and if she chooses Society, she gets $u_2(x_1, 'Society') = x_{11} - x_{12}$.

Zero-sum Games

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Column plays best response: Column should get

$$\max\{-3x_{11} + 2x_{12}, x_{11} - x_{12}\}.$$

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If row wants to maximize her utility, she should play then

$$(x_{11}^*, x_{12}^*) = \arg \max_{x_{11}, x_{12}} \min\{3x_{11} - 2x_{12}, -x_{11} + x_{12}\}$$

Zero-sum Games

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Linear Program for Row player

$$\begin{aligned} & \max z \\ \text{s.t. } & 3x_{11} - 2x_{12} \geq z \\ & -x_{11} + x_{12} \geq z \\ & x_{11} + x_{12} = 1 \\ & x_{11}, x_{12} \geq 0 \end{aligned}$$

Zero-sum Games

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$$\text{Sol } x_1 = (\frac{3}{7}, \frac{4}{7}), z = \frac{1}{7}$$

Row gets at least 1/7!

Zero-sum Games

Suppose now that column player plays (x_{21}, x_{22}) . How should row player respond?

Answer: If she chooses Economy she gets in expectation $u_1('Economy', x_2) = 3x_{21} - x_{22}$ and if she chooses Education, she gets $u_1('Education', x_2) = -2x_{21} + x_{22}$.

Row plays best response: Row should get

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Answer: If she chooses Economy she gets in expectation $u_1('Economy', x_2) = 3x_{21} - x_{22}$ and if she chooses Education, she gets $u_1('Education', x_2) = -2x_{21} + x_{22}$.

Row plays best response: Row should get

$$\max\{3x_{21} - x_{22}, -2x_{21} + x_{22}\}.$$

Column gets (zero-sum):

$$\min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}.$$

If Column wants to maximize her utility, she should play then

$$(x_{21}^*, x_{22}^*) = \arg \max_{x_{21}, x_{22}} \min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$$

Zero-sum Games

Suppose now that column player plays (x_{21}, x_{22}) . How should row player respond?

Answer: If she chooses Economy she gets in expectation $u_1('Economy', x_2) = 3x_{21} - x_{22}$ and if she chooses Education, she gets $u_1('Education', x_2) = -2x_{21} + x_{22}$.

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**Linear Program
for column player**

$$\begin{array}{ll}\max & w \\ \text{s.t.} & -3x_{21} + x_{22} \geq w \\ & 2x_{21} - x_{22} \geq w \\ & x_{21} + x_{22} = 1 \\ & x_{21}, x_{22} \geq 0\end{array}$$

If Column wants to maximize her utility, she should play then

$$(x_{21}^*, x_{22}^*) = \arg \max_{x_{21}, x_{22}} \min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$$

Zero-sum Games

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$$\min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}.$$

$$\text{Sol } x_2 = (\frac{2}{7}, \frac{5}{7}), w = -\frac{1}{7}$$

Column gets at least -1/7!

**Linear Program
for column player**

$$\begin{array}{ll}\max & w \\ \text{s.t.} & -3x_{21} + x_{22} \geq w \\ & 2x_{21} - x_{22} \geq w \\ & x_{21} + x_{22} = 1 \\ & x_{21}, x_{22} \geq 0\end{array}$$

If Column wants to maximize her utility, she should play then

$$(x_{21}^*, x_{22}^*) = \arg \max_{x_{21}, x_{22}} \min\{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$$

Zero-sum Games

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Linear Program for column player

$\max w$

Sol $x_2 = (\frac{2}{7}, \frac{5}{7}), w = -\frac{1}{7}$

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Since zero sum $(\frac{3}{7}, \frac{4}{7}), (\frac{2}{7}, \frac{5}{7})$ must be a Nash equilibrium!!

Next lecture LP duality to get NE and minimax theorem!

R

C

$$(x_{21}^*, x_{22}^*) = \arg \max_{x_{21}, x_{22}} \min \{-3x_{21} + x_{22}, 2x_{21} - x_{22}\}$$