

Diagnostic Midterm 1

Solutions: Problems 2–4

CS 169
Introduction to Optimization

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Problem 2: Taylor's theorem

For the following function, compute the first- and second-order approximations at $x_0 + h$.

$$f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2,$$

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1/10 \\ 1/5 \end{pmatrix}.$$

Give both expressions as functions of h , then substitute. Compare with the exact value and report absolute errors.

Value, gradient, and Hessian

$$f(x_0) = f(1, 0) = 1,$$

$$\nabla f(x) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}, \quad \nabla f(x_0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \nabla^2 f(x_0).$$

The mixed term x_1x_2 contributes 1 to **both** off-diagonal entries.

First-order approximation

$$f(x_0 + h) \approx f(x_0) + \nabla f(x_0)^\top h$$

$$= 1 + \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1 + 2h_1 + h_2.$$

At $h = (1/10, 1/5)$:

$$1 + 2 \left(\frac{1}{10} \right) + \frac{1}{5} = \frac{7}{5} = 1.4.$$

The second-order correction

$$\frac{1}{2}h^\top \nabla^2 f(x_0)h = \frac{1}{2} \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \frac{1}{2} \left(2h_1^2 + 2h_1h_2 + 2h_2^2 \right) = h_1^2 + h_1h_2 + h_2^2.$$

At the specified displacement, the correction is

$$\frac{1}{100} + \frac{1}{50} + \frac{1}{25} = \frac{7}{100} = 0.07.$$

Second-order approximation

$$f(x_0 + h) \approx f(x_0) + \nabla f(x_0)^\top h + \frac{1}{2} h^\top \nabla^2 f(x_0) h$$

$$= 1 + 2h_1 + h_2 + h_1^2 + h_1h_2 + h_2^2.$$

At $h = (1/10, 1/5)$:

$$\frac{7}{5} + \frac{7}{100} = \frac{147}{100} = 1.47.$$

Exact value and errors

$$x_0 + h = \left(\frac{11}{10}, \frac{1}{5} \right), \quad f(x_0 + h) = \frac{121 + 22 + 4}{100} = \frac{147}{100}.$$

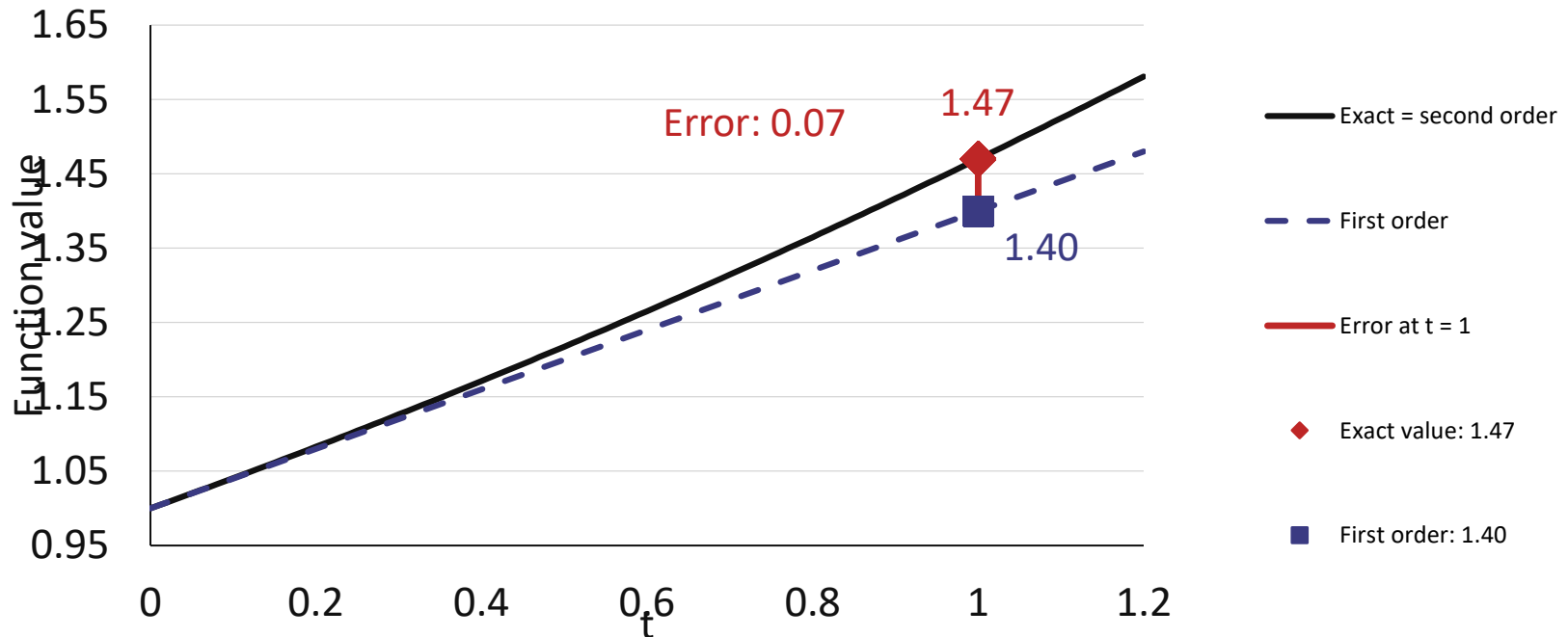
$$\text{First-order error: } \left| \frac{147}{100} - \frac{7}{5} \right| = \frac{7}{100},$$

$$\text{Second-order error: } \left| \frac{147}{100} - \frac{147}{100} \right| = 0.$$

The function is quadratic, so the second-order expression is **exact for every displacement** h .

Taylor error along the step

$$g(t) = f(x_0 + th) = 1 + \frac{2}{5}t + \frac{7}{100}t^2.$$



At $t = 1$, the first-order error is 0.07. The second-order curve coincides with g .

Problem 3: Convexity

Let $C, D \subseteq \mathbb{R}^n$ be convex. Prove that $C \cap D$ is convex.
Assume $C \cap D \neq \emptyset$.

Definition. A set contains the entire line segment between any two of its points if and only if it is convex.

$$u, v \in C \cap D, \quad \lambda \in [0, 1] \quad \implies \quad \lambda u + (1 - \lambda)v \in C \cap D.$$

Proof: Intersection of convex sets

Take arbitrary $u, v \in C \cap D$ and $\lambda \in [0, 1]$. Write $z = \lambda u + (1 - \lambda)v$.

Because $u, v \in C$ and C is convex, we have $z \in C$.

Because $u, v \in D$ and D is convex, we also have $z \in D$.

Thus $z \in C \cap D$. Since u, v and λ were arbitrary, $C \cap D$ is convex.

Problem 4: Production planning

Product 1 uses 1 hour of machine A and 3 hours of machine B. Its profit is \$3 per unit.

Product 2 uses 2 hours of machine A and 1 hour of machine B. Its profit is \$2 per unit.

Available time: 8 hours on machine A and 9 hours on machine B. **Fractional quantities are allowed.**

Formulate and solve the LP graphically, then write its dual.

Primal formulation

Let x_1, x_2 be the quantities of products 1 and 2.

$$\max \quad 3x_1 + 2x_2$$

$$\text{subject to} \quad x_1 + 2x_2 \leq 8 \quad (\text{machine A}),$$

$$3x_1 + x_2 \leq 9 \quad (\text{machine B}),$$

$$x_1, x_2 \geq 0.$$

Each resource constraint limits the **total time used** on that machine.

Vertices of the feasible region

On the axes, the resource constraints give the vertices $(0,0)$, $(3,0)$, and $(0,4)$.

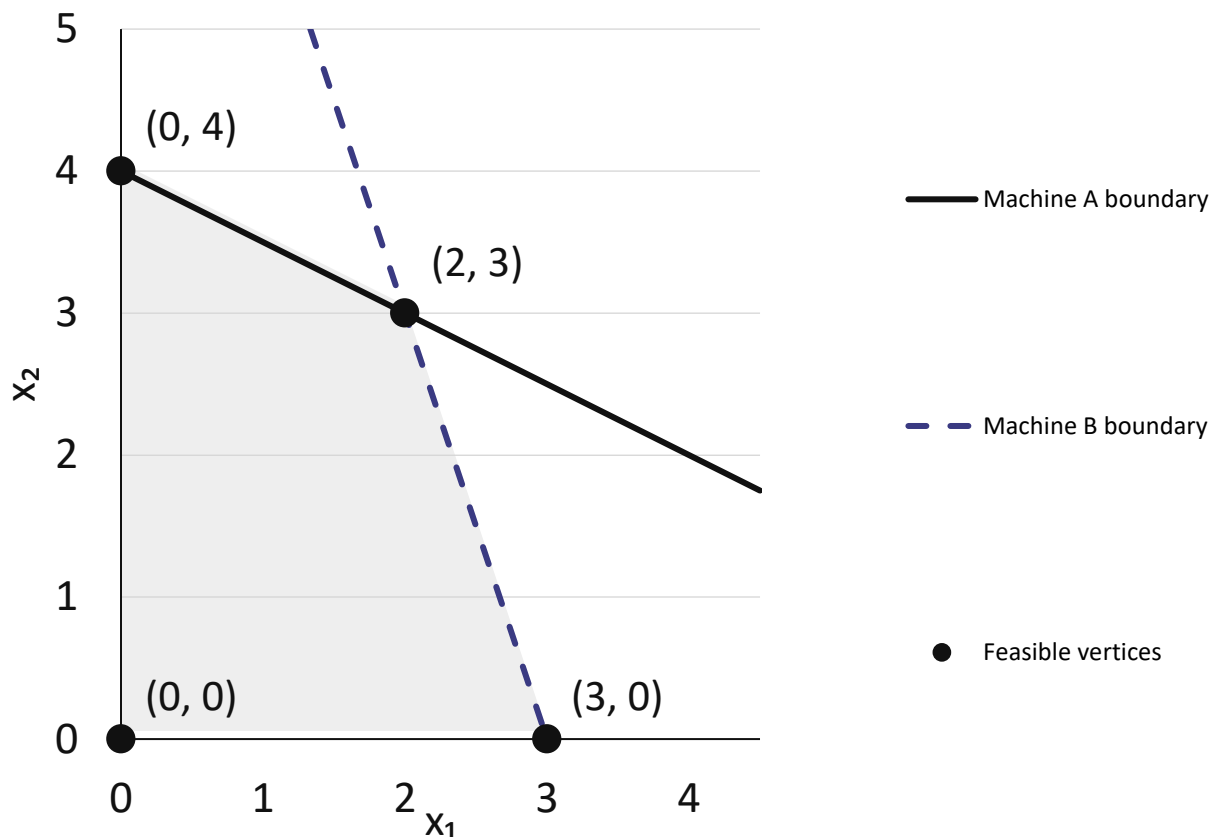
The fourth vertex is where both resource constraints are tight:

$$\begin{array}{lcl} x_1 + 2x_2 = 8, & \implies & 5x_2 = 15. \\ 3x_1 + x_2 = 9. & & \end{array}$$

$$x_2 = 3, \quad x_1 = 8 - 2(3) = 2.$$

The feasible region

The shaded region satisfies **all four inequalities**.



Objective values at the vertices

A linear objective on this bounded polygon attains its maximum at a vertex.

$$(0, 0) : 3(0) + 2(0) = 0,$$

$$(3, 0) : 3(3) + 2(0) = 9,$$

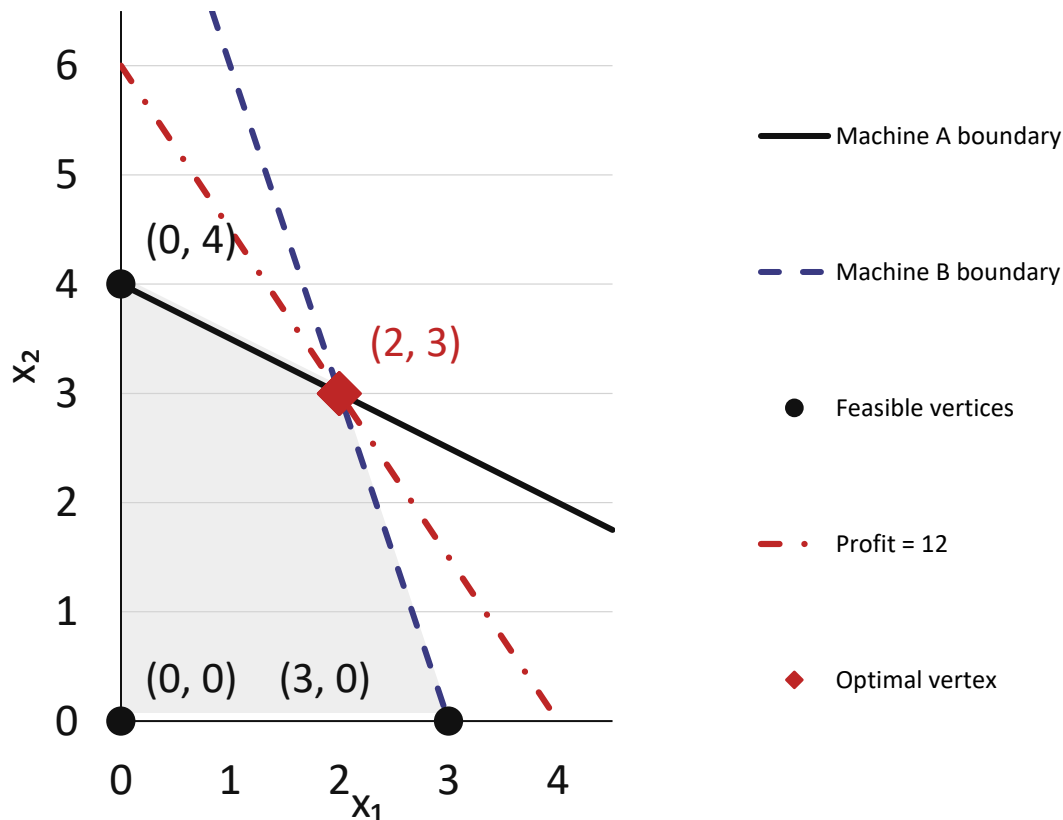
$$(2, 3) : 3(2) + 2(3) = 12,$$

$$(0, 4) : 3(0) + 2(4) = 8.$$

The largest value is 12, achieved at $(2, 3)$.

Optimal production and profit

The unique optimum is $(2, 3)$, with profit \$12.



How the dual is formed

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \quad c = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Assign $y_1 \geq 0$ to machine A and $y_2 \geq 0$ to machine B.
The dual minimizes $b^\top y$.

$$A^\top y \geq c \quad \Longleftrightarrow \quad \begin{cases} y_1 + 3y_2 \geq 3, \\ 2y_1 + y_2 \geq 2. \end{cases}$$

Each primal variable gives one dual constraint. The matrix is **transposed**.

Primal and dual

Primal: maximize profit

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 8, \\ & 3x_1 + x_2 \leq 9, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Dual: minimize cost

$$\begin{aligned} \min \quad & 8y_1 + 9y_2 \\ \text{s.t.} \quad & y_1 + 3y_2 \geq 3, \\ & 2y_1 + y_2 \geq 2, \\ & y_1, y_2 \geq 0. \end{aligned}$$

Resource limits become objective coefficients. Profit coefficients become constraint bounds.